

Sigma values of some graph families under join and corona operations

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ABSTRACT

Let $c: V(H) \rightarrow \mathbb{Z}^+$ be a vertex coloring of a simple nontrivial connected graph H . The color sum of a vertex $v \in V(H)$, denoted by $\sigma(v)$, is defined as the sum of the colors of the vertices adjacent to v . A coloring c of H is called a sigma coloring if $\sigma(u) \neq \sigma(v)$ for any two adjacent vertices u and v of H . The sigma chromatic number of H , denoted by $\sigma(H)$, is the minimum number of colors required in a sigma coloring of H . Let $\max(c)$ be the largest color assigned to a vertex of H by a coloring c . The sigma value of H , denoted by $\nu(H)$, is the minimum value of $\max(c)$ over all sigma ℓ -colorings c of H for which $\sigma(H) = \ell$. In this paper, we determine the sigma value of the join of paths or cycles and the corona of a cycle with a star graph. We also generalize results on the sigma value of the corona of a path with another graph whose sigma value is at least 2, or the corona of a cycle with another graph whose sigma value is at least 2.

INTRODUCTION

Graph coloring is a well-known branch of graph theory that researchers have studied over several decades. Recent studies on graph colorings, specifically sigma colorings, have explored the behavior of the sigma chromatic number of a graph when an operation is applied. We refer readers to the following papers for related studies on sigma colorings of graphs (Luzon et al. 2016), (Gonzaga and Almeida 2019), (Garciano et al. 2020), (Kumar and Pillai 2021), (Garciano et al. 2022), and (Yogalakshmi and Balamurugan 2022). Additional related studies on magic sigma coloring and sigma partitioning of graphs appear in (Swamy et al. 2021) and (Dehghan et al. 2018), respectively. In contrast, relatively few studies have focused on the sigma value of graphs.

While the sigma chromatic number determines the minimum number of colors needed for a sigma coloring, the sigma value provides additional information by determining how small the largest color can be among all sigma colorings. This makes the sigma value a parameter to study after the sigma chromatic number has been determined. Graph operations are also of particular interest in this setting because they change the neighborhoods of vertices in systematic ways and, consequently, affect the neighborhood color sums that define a sigma coloring. Motivated by this gap, we investigate the sigma value of the join of paths or cycles and the corona of a cycle with a star graph. We further establish general results for coronas involving paths or cycles and graphs satisfying suitable conditions on their sigma colorings. These results extend the existing theory on sigma values and provide constructions and conditions that may be useful in determining the sigma value of other graph families obtained through similar operations.

We define some graph-theoretic concepts introduced by Chartrand, Okamoto, and Zhang (2010). Given a graph H , let $c: V(H) \rightarrow \mathbb{Z}^+$ be a vertex coloring of a simple nontrivial connected graph H , where neighboring vertices are allowed to receive identical colors. The color sum $\sigma(v)$ of a vertex v in $V(H)$ is defined as the sum of colors assigned to the neighbors of v . If $\sigma(u) \neq \sigma(v)$, for every pair of adjacent vertices u and v in H , then c is called a sigma coloring of H . The minimum number of colors required in a sigma coloring of H is called the sigma chromatic number of H , denoted by $\sigma(H)$. Consequently, a sigma coloring induces a proper coloring of H in which each vertex v is assigned the color sum $\sigma(v)$. Unless otherwise stated, all graphs in this study are nontrivial, connected, and undirected.

Figure 1 presents sigma colorings of a graph G_1 with order $|V(G_1)| = 6$ and size $|E(G_1)| = 7$. Figure 1(a) illustrates a sigma coloring of G_1 using three colors where the color sums are

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displayed near each vertex using the notation σ_i (for $1 \leq i \leq 6$) for brevity. Figure 1(b) illustrates that G_1 also admits a sigma coloring using only two colors. Since vertices u and v are adjacent and have the same degree. Therefore, assigning one color to all vertices of G_1 is not feasible. Such an assignment would not satisfy

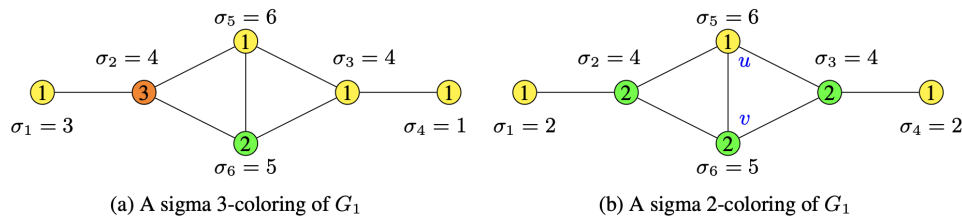


Figure 1: Sigma colorings of G_1 with $|V(G_1)| = 6$ and $|E(G_1)| = 7$.

For a graph H , with $\chi(H) = \ell$, there exists a proper coloring of H using the colors $1, 2, \dots, \ell$. While other sets of colors are possible, ℓ is the minimum among the largest colors across all proper colorings of H . In the context of sigma colorings, however, this property does not necessarily hold. Suppose $\sigma(H) = \ell$ and let c be a sigma coloring of H with ℓ colors. Denote $\max(c) = \max\{c(v) | v \in V(H)\}$. The sigma value of H is defined by $v(H) = \min\{\max(c) | c \text{ is a sigma } \ell\text{-coloring of } H\}$.

Zhang (2016) showed that $\sigma(G) \leq v(G)$. Bulay-og et al. (2021) proved that the sigma value of the join of a finite number of even cycles of the same order is 2. This paper determines the sigma value of the join and corona of some known families of graphs. These graph operations were introduced in (Harary 1969). Let H_1 and H_2 be disjoint graphs and let $|V(H_1)| = n_1$ and $|V(H_2)| = n_2$. The join of H_1 and H_2 , denoted by $H_1 + H_2$, is a graph whose vertex set $V(H_1 + H_2)$ is the union of $V(H_1)$ and $V(H_2)$ and whose edge set is $E(H_1 + H_2) = \{uv : u \in V(H_1), v \in V(H_2)\} \cup E(H_1) \cup E(H_2)$. On the other hand, the corona of H_1 and H_2 , denoted by $H_1 \odot H_2$, is a graph obtained by taking one copy of H_1 and n_1 copies of H_2 , and then joining the i th vertex of H_1 with an edge to every vertex in the i th copy of H_2 .

Preliminaries

The following previously established results will be used in this paper. These results serve as fundamental tools and provide the background needed for the main results. Remark 1 provides a characterization of connected graphs with sigma chromatic number equal to 1 by relating this to the degrees of neighboring vertices.

Remark 1 (Chartrand et al. 2010). Let G be a nontrivial connected graph. Then $\sigma(G) = 1$ if and only if any two adjacent vertices of G have different degrees.

Previous work has determined the sigma chromatic numbers of some known families of graphs such as star graphs, paths, and cycles, as established by Chartrand et al. (2010). We recall these results as follows:

Lemma 2 (Chartrand et al. 2010). Let S_n be a star of order $n \geq 3$. Then, $\sigma(S_n) = 1$.

Theorem 3 (Chartrand et al. 2010). Let P_n be a path of order n , where $n \geq 1$. Then, $\sigma(P_n) = v(P_n) = 1$ when $n \in \{1, 3\}$, otherwise, $\sigma(P_n) = v(P_n) = 2$.

Theorem 4 (Chartrand et al. 2010). If C_n is a cycle of order n , where $n \geq 3$, then $\sigma(C_n) = v(C_n) = 2$ when n is even and $\sigma(C_n) = v(C_n) = 3$ when n is odd.

Corollary 5 characterizes graphs whose sigma chromatic number is equal to its order, while Lemma 6 describes how sigma colorings of disjoint graphs can be combined to form a coloring of the resulting corona graph.

Corollary 5 (Chartrand et al. 2010). Let G be a nontrivial connected graph of order n . Then $\sigma(G) = n$ if and only if $G = K_n$.

the definition of a sigma coloring. Observe that any two adjacent vertices have distinct color sums. Thus, the sigma chromatic number of G_1 is given by $\sigma(G_1) = 2$.

Lemma 6 (Garciano et al. 2021). Let c_1 and c_2 be sigma colorings of disjoint graphs G and H , respectively. Define a coloring $c = (c_1, c_2)$ to be a coloring of $G \odot H$ such that G is colored using c_1 and every copy of H is colored using c_2 . If u and v are adjacent vertices that are both in G or both in H , then $\sigma_c(u) \neq \sigma_c(v)$.

The following remark establishes a lower bound for the sigma chromatic number of the corona of graphs in terms of the sigma chromatic number of one of its factor graphs. In a sigma coloring of a corona graph, the restriction to each attached copy must itself remain a valid sigma coloring, thus yielding a natural lower bound on the sigma chromatic number of the resulting graph. This idea is presented in the following remark.

Remark 7 (Chartrand et al. 2010). Let c be a sigma coloring on $G \odot H$, where G and H are disjoint graphs. Then the restriction of c on any copy of H is a sigma coloring. Hence, $\sigma(G \odot H) \geq \sigma(H)$.

A natural question is how small the sigma value of a graph can be. The next result presents a useful relationship between the sigma chromatic number of G and the sigma value of G .

Theorem 8 (Chartrand et al. 2010). For a nontrivial connected graph G , $\sigma(G) \leq v(G)$.

This inequality guides several arguments in the succeeding results.

MAIN RESULTS

We now present the main contributions of this paper. The main results are divided into two sections: the first investigates the sigma value of the join of paths or cycles, while the second examines the sigma value of the corona of a graph G with paths, cycles, or star graphs. These results extend previous findings and establish new properties of sigma colorings of these graph constructions.

Sigma value of the join of paths or cycles

In this section, we characterize the sigma value of the join of two paths P_n and P_m , where $6 \leq n \leq m$, and the join of two cycles C_n and C_m , where $4 \leq n \leq m$. Specifically, we determine the minimum $k > 0$ such that there exists a sigma k -coloring.

For all $n \geq 4$, let $V(P_n) = \{v_1, v_2, \dots, v_n\}$ such that $v_i v_{i+1} \in E(P_n)$ for $1 \leq i \leq n-1$. By Theorem 3, $v(P_n) = 2$. Guided by this result, we construct four distinct sigma 2-colorings of P_n using the colors 1 and 2 that yield the minimum or maximum possible total color sum of the vertices of P_n according to the congruence class of n modulo 4. Accordingly, we consider two cases.

Case 1: Assume that $n \not\equiv 1 \pmod{4}$. Define a coloring c_1 on the vertices of P_n as: $c_1(v_i) = 2$ if $4|i$; $c_1(v_i) = 1$, otherwise. We illustrate in Figure 2 a sigma 2-coloring c_1 of P_n , when $n \not\equiv$

1 (mod 4). In this construction, the vertices of the path are colored according to the repeating color sequence (1,1,1,2), which determines the pattern of colors assigned along the vertices of P_n .

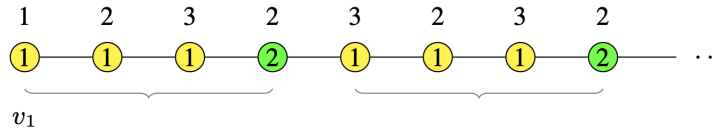


Figure 2: Sigma 2-coloring c_1 of P_n , where $n \not\equiv 1 \pmod{4}$

Then, the color sums of the vertices in P_n are given by

$$\sigma_{c_1}(v_i) = \begin{cases} 1, & \text{if } i = 1 \text{ or } i = n, \\ 2, & \text{if } i \text{ is even and } 1 < i < n, \\ 3, & \text{if } i \text{ is odd and } 1 < i < n. \end{cases}$$

Case 2: Suppose $n \equiv 1 \pmod{4}$. Define a coloring c_2 on the vertices of P_n as: $c_2(v_i) = 2$, when $i \equiv 3 \pmod{4}$, and $c_2(v_i) = 1$, otherwise. Figure 3 illustrates a sigma 2-coloring c_2 of P_n , when $n \equiv 1 \pmod{4}$. Observe that the vertices of the path are colored according to the repeating color sequence (1,1,2,1), which determines the pattern of colors assigned along the vertices of P_n .

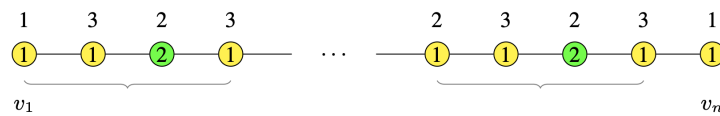


Figure 3: Sigma 2-coloring c_2 of P_n , where $n \equiv 1 \pmod{4}$

Then, the color sums of the vertices in P_n are as follows:

$$\sigma_{c_2}(v_i) = \begin{cases} 1, & \text{if } i = 1 \text{ or } i = n, \\ 2, & \text{if } i \text{ is odd and } 1 < i < n, \\ 3, & \text{if } i \text{ is even and } 1 < i < n. \end{cases}$$

It is straightforward to verify that c_1 and c_2 are sigma 2-colorings of P_n using the colors 1 and 2. Now, note that in any sigma 2-coloring of P_n , it is not possible for four consecutive vertices to have the same color, as this would violate the definition of a sigma coloring. Indeed, suppose u_1, u_2, u_3 and u_4 are four consecutive vertices of P_n , and suppose the same color, say a , is assigned to each of these vertices, then $\sigma(u_2) = \sigma(u_3) = 2a$. By construction, the colorings c_1 and c_2 maximize the frequency with which the smallest color, 1, is assigned to the vertices in P_n , minimizing the number of vertices assigned the largest color, 2. Let $S = \sum_{x \in V(G)} c(x)$ where c is a sigma coloring of G . A sigma coloring of G is called a minimal sigma coloring if the corresponding value of S is the smallest possible total color sum among all sigma colorings using a prescribed set of colors. Conversely, a maximal sigma coloring of G is one that produces the largest value of S among all sigma colorings using the same prescribed set of colors. In this case, the colorings c_1 and c_2 are minimal sigma 2-colorings of P_n using the colors 1 and 2. A maximal sigma 2-coloring of P_n using the colors 1 and 2 can be constructed as follows:

Define a coloring c_3 on the vertices of P_n as: $c_3(v_i) = 1$ if $n \equiv 0 \pmod{4}$, and $c_3(v_i) = 2$ if $n \not\equiv 0 \pmod{4}$. If $n \not\equiv 1 \pmod{4}$, then the color sums of the vertices in P_n are given by

$$\sigma_{c_3}(v_i) = \begin{cases} 2, & \text{if } i \in \{1, n\}, \\ 3, & \text{if } i \text{ is odd and } 3 \leq i < n, \\ 4, & \text{if } i \text{ is even and } 2 \leq i < n. \end{cases}$$

On the other hand, if $n \equiv 1 \pmod{4}$, then the color sums of the vertices in P_n are given by

$$\sigma_{c_3}(v_i) = \begin{cases} 1, & \text{if } i = n, \\ 2, & \text{if } i = 1, \\ 3, & \text{if } i \text{ is odd and } 3 \leq i < n, \\ 4, & \text{if } i \text{ is even and } 2 \leq i < n. \end{cases}$$

Since any two adjacent vertices in P_n have distinct color sums with respect to the coloring c_3 , we say that c_3 is a sigma coloring of P_n . Moreover, since the largest color 2 is most often used, c_3 is a maximal sigma coloring of P_n . Let $\mu(n)$ and $M(n)$ denote, respectively, the sum of vertex colors of P_n under minimal and maximal sigma 2-colorings using the colors 1 and 2. In particular, $\mu(n) = n + \lfloor \frac{n}{4} \rfloor$ and $M(n) = 2n - \lfloor \frac{n}{4} \rfloor$. If $m \geq n$, then $M(m) - \mu(n) \geq M(n) - \mu(n) \geq n - 2\lfloor \frac{n}{4} \rfloor$.

We now present Theorem 9 as follows.

Theorem 9. Let P_n and P_m be paths of orders n and m , respectively, such that $6 \leq n \leq m$. Then, $v(P_n + P_m) = 2$.

Proof: We divide the proof into two cases.

Case 1: Suppose $n \not\equiv 1 \pmod{4}$. Let $v \in V(P_n + P_m)$. Define a coloring c^1 of $P_n + P_m$ as: $c^1(v) = c_1(v)$ if $v \in V(P_n)$, and $c^1(v) = c_3(v)$ if $v \in V(P_m)$. Then, the color sum of each vertex v is as follows:

$$\sigma_{c^1}(v) = \begin{cases} \sigma_{c_1}(v) + M(m), & \text{if } v \in V(P_n) \\ \sigma_{c_3}(v) + \mu(n), & \text{if } v \in V(P_m). \end{cases}$$

Note that if x and y are both vertices in P_n or in P_m , then $\sigma_{c^1}(x) \neq \sigma_{c^1}(y)$. Let $x \in V(P_n)$ and $y \in V(P_m)$. Note that $\sigma_{c_1}(x) \geq 1$ and $\sigma_{c_3}(y) \leq 4$. Since the sequences $\{M(n)\}$ and $\{\mu(n)\}$ are increasing, $m \geq n \geq 4$, and $(n, m) \notin \{(4, 4), (5, 5)\}$, it follows that $M(m) - \mu(n) \geq 4$. Thus, $\sigma_{c^1}(x) = \sigma_{c_1}(x) + M(m) \geq \sigma_{c_1}(x) + 4 + \mu(n) \geq 5 + \mu(n) > \sigma_{c_3}(y) + \mu(n) = \sigma_{c^1}(y)$ and so, c^1 is a sigma 2-coloring of $P_n + P_m$ using the colors 1 and 2.

Case 2: Suppose $n \equiv 1 \pmod{4}$. Then, $n \geq 9$. Let v be any vertex of the join $P_n + P_m$. Define a coloring c^2 of $P_n + P_m$ as: $c^2(v) = c_2(v)$ if $v \in V(P_n)$, and $c^2(v) = c_3(v)$ if $v \in V(P_m)$. Then, the color sum of v is given as follows:

$$\sigma_{c^2}(v) = \begin{cases} \sigma_{c_2}(v) + M(m), & \text{if } v \in V(P_n) \\ \sigma_{c_3}(v) + \mu(n), & \text{if } v \in V(P_m). \end{cases}$$

As in Case 1, let $x \in V(P_n)$ and $y \in V(P_m)$. Then, $\sigma_{c_2}(x) \geq 1$ and $\sigma_{c_3}(y) \leq 4$. Since $M(m) \geq M(n)$, we obtain $M(m) \geq 5 + \mu(n)$. Thus, $\sigma_{c^2}(x) = \sigma_{c_2}(x) + M(m) \geq \sigma_{c_2}(x) + 5 + \mu(n) \geq 6 + \mu(n) > \sigma_{c_3}(y) + \mu(n) = \sigma_{c^2}(y)$.

Hence, c_2 is a sigma 2-coloring of $P_n + P_m$ using the colors 1 and 2.

■

We provide an example to illustrate the application of Theorem 9. Figure 4 shows a sigma 2-coloring of the graph $P_6 + P_7$ using the colors 1 and 2. Note that the coloring is constructed using the coloring c^1 described in Case 1 of the proof of Theorem 9. The numbers shown outside the vertices are the corresponding color sums.

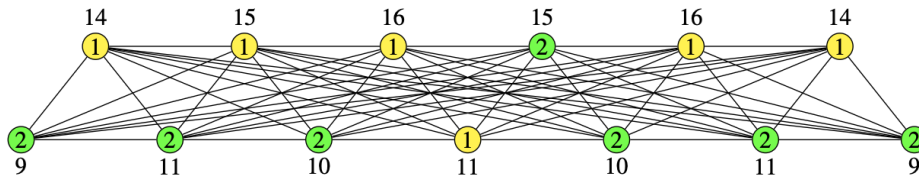


Figure 4: Sigma 2-coloring of $P_6 + P_7$

Consider now a cycle C_n , where $n \geq 3$. Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and $E(C_n) = \{v_i v_{i+1} | 1 \leq i < n\} \cup \{v_1 v_n\}$. By Theorem 4, $\nu(C_n) = 2$ if n is even and $\nu(C_n) = 3$ if n is odd. We present sigma colorings of C_n using the colors 1 and 2 if n is even and using the colors 1, 2, and 3 if n is odd.

Case 1: Suppose n is even. Define a coloring c_4 on the vertices of C_n using the colors 1 and 2 by

$$c_4(v_i) = \begin{cases} 2, & \text{if } 4 \mid i \text{ or } i = n, \\ 1, & \text{otherwise.} \end{cases}$$

Then, the color sums of the vertices in C_n are as follows: when $n \equiv 0 \pmod{4}$,

$$\sigma_{c_4}(v_i) = \begin{cases} 2, & \text{if } i \text{ is even} \\ 3, & \text{if } i \text{ is odd} \end{cases}$$

and when $n \equiv 2 \pmod{4}$,

$$\sigma_{c_4}(v_i) = \begin{cases} 2, & \text{if } i \text{ is even} \\ 3, & \text{if } i \text{ is odd and } i \neq n-1 \\ 4, & \text{if } i = n-1. \end{cases}$$

Figure 5 depicts the sigma 2-coloring c_4 of C_n , showing the pattern of colors assigned to its vertices. The numbers shown above the vertices are the corresponding color sums.

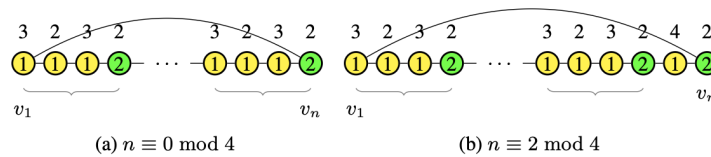


Figure 5: Minimal sigma 2-coloring c_4 of C_n , where n is even

Case 2: Suppose n is odd. Define a coloring c_5 on the vertices of C_n using the colors 1, 2, and 3 by

$$c_5(v_i) = \begin{cases} 2, & \text{if } 4 \mid i \\ 3, & \text{if } i = n \\ 1, & \text{otherwise.} \end{cases}$$

and when $n \equiv 3 \pmod{4}$,

$$\sigma_{c_5}(v_i) = \begin{cases} 2, & \text{if } i = n, \text{ or } i \text{ is even and } 1 \leq i \leq n-3 \\ 3, & \text{if } i \text{ is odd and } 3 \leq i \leq n-2 \\ 4, & \text{if } i \in \{1, n-1\}. \end{cases}$$

Then, the color sums of the vertices in C_n are as follows: when $n \equiv 1 \pmod{4}$,

$$\sigma_{c_5}(v_i) = \begin{cases} 2, & \text{if } i \text{ is even and } 2 \leq i \leq n-3 \\ 3, & \text{if } i \text{ is odd and } 3 \leq i \leq n \\ 4, & \text{if } i \in \{1, n-1\}; \end{cases}$$

To visualize the construction, the coloring c_5 of C_n is illustrated in Figure 6.

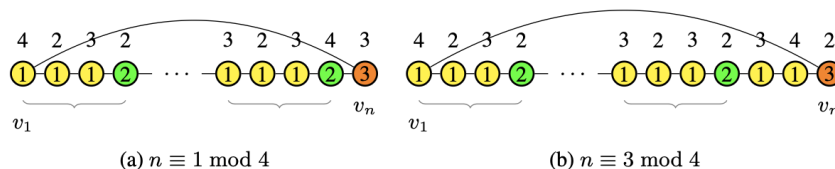


Figure 6: Minimal sigma 3-coloring c_5 of C_n , where n is odd

It can be easily checked that c_4 is sigma 2-coloring of C_n using the colors 1 and 2 and that c_5 is sigma 3-coloring of C_n using the colors 1, 2, and 3. As in the case of paths, in any sigma coloring of C_n , it is not possible for four consecutive vertices to have the same color. From the definition, the coloring c_4 maximizes the number

of times that the smallest color, 1, is assigned to the vertices of C_n and minimizes the number of times that the largest color, 2, is assigned. That is, in every four consecutive vertices, three of them are colored 1 and the remaining vertex is colored 2. In the case of odd cycles, c_5 maximizes the number of times that the smallest

color, 1, is assigned to the vertices of C_n and minimizes the number of times that the largest color, 3, is assigned. Thus, c_4 is a minimal sigma 2-coloring of C_n (where n is even) using the colors 1 and 2 while c_5 is a minimal sigma 3-coloring of C_n (where n is odd) using the colors 1, 2, and 3. By interchanging the colors 1 and 2 in the coloring c_4 , or 1 and 3 in the coloring c_5 , maximal sigma colorings of C_n using colors 1 and 2 if n is even or 1, 2, and 3 if n is odd can be obtained. Let us denote these colorings by c_6 and c_7 , respectively.

Let $\mu(n)$ and $M(n)$ denote the sum of colors in a minimal and maximal sigma coloring of C_n , respectively, using the set of colors $\{1, 2\}$ if n is even and $\{1, 2, 3\}$ if n is odd. Then, the values of $\mu(n)$ and $M(n)$ are given by

$$\mu(n) = \begin{cases} 5 \left\lfloor \frac{n}{4} \right\rfloor, & \text{if } n \equiv 0 \pmod{4} \\ 5 \left\lfloor \frac{n}{4} \right\rfloor + 3, & \text{if } n \equiv 1 \text{ or } 2 \pmod{4} \\ 5 \left\lfloor \frac{n}{4} \right\rfloor + 5, & \text{if } n \equiv 3 \pmod{4}, \end{cases}$$

and

$$M(n) = \begin{cases} 7 \left\lfloor \frac{n}{4} \right\rfloor, & \text{if } n \equiv 0 \pmod{4} \\ 7 \left\lfloor \frac{n}{4} \right\rfloor + 3, & \text{if } n \equiv 2 \pmod{4} \\ 11 \left\lfloor \frac{n}{4} \right\rfloor + 1, & \text{if } n \equiv 1 \pmod{4} \\ 11 \left\lfloor \frac{n}{4} \right\rfloor + 7, & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

Now, let $q \geq n \geq 4$. Then, $M(q) - \mu(n) \geq M(q) - \mu(q) \geq 2 \left\lfloor \frac{q}{4} \right\rfloor \geq 2$.

With the preceding discussion in place, we now present Theorem 10, which establishes a key result.

Theorem 10. Let C_n and C_m be cycles of orders n and m , respectively, such that $4 \leq n \leq m$ and $(n, m) \notin \{(4, 4), (5, 5), (6, 6)\}$. Then, $v(C_n + C_m) = 2$ when both n and m are even and $v(C_n + C_m) = 3$ when at least one of n or m is odd.

Proof: We divide the proof into two cases.

Case 1: Suppose both n and m are even, where $4 \leq n \leq m$ and $(n, m) \notin \{(4, 4), (5, 5), (6, 6)\}$. Let v be any vertex of the join $C_n + C_m$. Define a coloring c^3 of $C_n + C_m$ using the colors 1 and 2 by

$$c^3(v) = \begin{cases} c_4(v), & \text{if } v \in V(C_n) \\ c_6(v), & \text{if } v \in V(C_m). \end{cases}$$

Then, the color sum of v is as follows:

$$\sigma_{c^3}(v) = \begin{cases} \sigma_{c_4}(v) + M(m), & \text{if } v \in V(C_n) \\ \sigma_{c_6}(v) + \mu(n), & \text{if } v \in V(C_m), \end{cases}$$

Let $x \in V(C_n)$ and $y \in V(C_m)$. Note that $\sigma_{c_4}(x) \geq 2$ and $\sigma_{c_6}(y) \leq 4$. Since $m \geq n \geq 4$ and $(n, m) \notin \{(4, 4), (6, 6)\}$, we obtain $M(m) \geq 4 + \mu(n)$. Thus,

$$\begin{aligned} \sigma_{c^3}(x) &= \sigma_{c_4}(x) + M(m) \geq \sigma_{c_4}(x) + 4 + \mu(n) \geq 6 + \mu(n) \\ &> 4 + \mu(n) \geq \sigma_{c_6}(y) + \mu(n) = \sigma_{c^3}(y). \end{aligned}$$

Hence, c^3 is a sigma 2-coloring of $C_n + C_m$ using the colors 1 and 2.

Case 2: Suppose at least one of n or m is odd, where $4 \leq n \leq m$ and $(n, m) \neq (5, 5)$. Let c_M be a sigma 2-coloring of C_q , where q is even, using the colors 2 and 3 such that for each $1 \leq i \leq q$, $c(v_i) = 2$ if $i \equiv 0 \pmod{4}$ or $i = q$, and $c(v_i) = 3$, otherwise. Define a coloring c^4 of $C_n + C_m$ using the colors 1, 2, and 3 by

$$c^4(v) = \begin{cases} c_4(v), & \text{if } v \in V(C_n) \text{ and } n \text{ is even,} \\ c_5(v), & \text{if } v \in V(C_n) \text{ and } n \text{ is odd,} \\ c_7(v), & \text{if } v \in V(C_m) \text{ and } m \text{ is odd,} \\ c_M(v), & \text{if } v \in V(C_m) \text{ and } m \text{ is even.} \end{cases}$$

Equivalently, the coloring c^4 assigns a minimal coloring to the vertices of C_n while the vertices of C_m are assigned a maximal coloring that uses at least two colors from the set $\{1, 2, 3\}$. For each vertex $v \in V(C_n + C_m)$, the color sum of v is

$$\sigma_{c^4}(v) = \begin{cases} \sigma_{c_4}(v) + M_3(m), & \text{if } v \in V(C_n) \text{ and } n \text{ is even} \\ \sigma_{c_5}(v) + M_3(m), & \text{if } v \in V(C_n) \text{ and } n \text{ is odd} \\ \sigma_{c_7}(v) + \mu(n), & \text{if } v \in V(C_m) \text{ and } m \text{ is odd} \\ \sigma_{c_M}(v) + \mu(n), & \text{if } v \in V(C_m) \text{ and } m \text{ is even,} \end{cases}$$

where $M_3(m) = \sum_{u \in V(C_m)} c_7(u)$ if m is odd, or $M_3(m) = \sum_{u \in V(C_m)} c_M(u)$ if m is even. Notice that $M_3(n) = M(n)$ if n is odd. On the other hand, if n is even and $n \equiv 0 \pmod{4}$ then $M_3(n) = 11 \left\lfloor \frac{n}{4} \right\rfloor$, and if $n \equiv 2 \pmod{4}$ then $M_3(n) = 11 \left\lfloor \frac{n}{4} \right\rfloor + 5$. Now, let $x \in V(C_n)$ and $y \in V(C_m)$. Note that $\sigma_{c_4}(x) \geq 2$, $\sigma_{c_5}(x) \geq 2$, $\sigma_{c_M}(y) \leq 6$, and $\sigma_{c_7}(y) \leq 6$. Recall that $4 \leq n \leq m$ and $(n, m) \notin \{(4, 4), (5, 5), (6, 6)\}$. Since $\{M_3(n)\}$ is non-decreasing, we have $M_3(m) - \mu(n) \geq M_3(n) - \mu(n) \geq 6$. Hence, $\sigma_{c^4}(x) \geq 2 + M_3(m) \geq 8 + \mu(n) > 6 + \mu(n) \geq \sigma_{c^4}(y)$. Thus, c^4 is a sigma coloring of $C_n + C_m$ using the colors 1, 2, and 3.

Combining the two cases yields $v(C_n + C_m) = 2$ when both n and m are even, and $v(C_n + C_m) = 3$ otherwise. ■

Note that the cases $(n, m) \in \{(4, 4), (5, 5), (6, 6)\}$ are excluded from Theorem 10 because the conditions used to distinguish the color sums of vertices from C_n and C_m do not hold for these pairs.

Figure 7 shows a sigma 3-coloring of the join $C_6 + C_7$ using the colors 1, 2, and 3. Note that the coloring is constructed using the coloring c^4 described in Case 2 of the proof of Theorem 10. The numbers shown outside the vertices are the corresponding color sums.

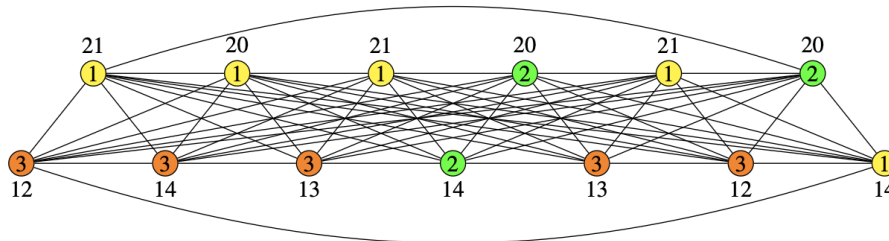


Figure 7: Sigma 3-coloring of $C_6 + C_7$

Sigma value of the corona of a graph G with paths, cycles, or star graphs

In this section, we investigate the sigma value of the corona graphs formed by some known families of graphs. We examine the sigma value of the corona of a path with an arbitrary graph G , as well as the corona of a cycle C_n with a graph G . In addition, we explicitly determine the sigma value of the corona of a cycle C_n with a star graph S_m as a notable special case. These results highlight how the structure of the base graphs influences the behavior of the sigma coloring of their corona graphs.

We now determine the sigma value of the corona of a nontrivial simple connected graph G with a path P_n under the condition that $\nu(G) = 2$.

Theorem 11. Let G be a nontrivial simple connected graph such that $\nu(G) = 2$ and let P_n be a path of order n , where $n \geq 4$. Then, $\nu(G \odot P_n) = 2$.

Proof: By Remark 7 and by Theorem 3, $\sigma(G \odot P_n) \geq 2$. Since $\nu(G) = 2$, there exists a sigma 2-coloring c_G using the colors 1 and 2. To show that $\nu(G \odot P_n) = 2$, we exhibit a sigma 2-coloring of $G \odot P_n$ using the colors 1 and 2. Let $v \in V(G \odot P_n)$. Define a coloring c of $G \odot P_n$ to be $c(v) = c_3(v)$ when $v \in V(P_n)$, where c_3 is the maximal coloring of P_n defined in the previous section and is applied to each copy of P_n and $c(v) = c_G(v)$ when $v \in V(G)$. Note that the restricted colorings $c|_G$ and $c|_{P_n}$ are sigma colorings of G and P_n , respectively. Hence, we are left to show that $\sigma(x) \neq \sigma(y)$ for any $x \in V(G)$ and $y \in V(P_n)$ such that $xy \in E(G \odot P_n)$. Now, observe that $\sigma(x) = \sigma_G(x) + S_n$, where $S_n = \sum_{v \in V(P_n)} c|_{P_n}(v)$, and $\sigma(y) = \sigma_{P_n}(y) + c(x)$. Since $\sigma_G(x) \geq 1$, $\sigma_{P_n}(y) \leq 4$, $c(x) \in \{1, 2\}$, and $S_n \geq 7$, it follows that $\sigma(x) \geq 1 + 7 = 8 > 6 \geq \sigma(y)$. Thus, c is a sigma 2-coloring of $G \odot P_n$ using the colors 1 and 2. Consequently, $\nu(G \odot P_n) = 2$. ■

The following corollary follows directly from the preceding theorem by considering the case where $G = P_n$.

Corollary 12. Let $n, m \geq 4$. Then, $\nu(P_n \odot P_m) = 2$.

Next, we determine the sigma value of the corona of a nontrivial simple connected graph G with a cycle C_n , distinguishing between the cases when n is even and when n is odd.

Theorem 13. Let G be a nontrivial simple connected graph such that $\nu(G) = 2$, and let C_n be a cycle of order n , where $n \geq 3$. Then,

$$\nu(G \odot C_n) = \begin{cases} 2, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$$

Proof: We consider two cases according to the parity of n . When n is even, the argument follows analogously to the proof of the first part of Theorem 11, except that the minimal sigma coloring c_4 defined in the previous section is used to color the vertices of C_n . Hence, it remains to establish the result for the case when n is odd.

Suppose n is odd. By Remark 7 and by Theorem 4, $\nu(G \odot C_n) \geq 3$. We now show that there exists a sigma 3-coloring of $G \odot C_n$ using the colors 1, 2, and 3. Let $v \in V(G \odot C_n)$. Define a coloring c of $G \odot C_n$ as follows:

$$c(v) = \begin{cases} c_5(v), & \text{if } v \in V(C_n) \\ c_G(v), & \text{if } v \in V(G), \end{cases}$$

where c_5 is a minimal coloring, defined in the previous section, which is applied to each copy of C_n and c_G is a minimal sigma coloring of G using the colors 1 and 2. Note that the restricted

colorings $c|_G$ and $c|_{C_n}$ are sigma colorings of G and C_n , respectively, and so, we are left to show that $\sigma(x) \neq \sigma(y)$ for any $x \in V(G)$ and $y \in V(C_n)$ such that $xy \in E(G \odot C_n)$. Now, observe that $\sigma(x) = \sigma_G(x) + S_n$ and $\sigma(y) = \sigma_{C_n}(y) + c(x)$. Since $\sigma_G(x) \geq 1$, $\sigma_{C_n}(y) \leq 4$, $c(x) \in \{1, 2, 3\}$, and $S_n \geq 8$, it follows that $\sigma(x) \geq 1 + 8 = 9 > 7 \geq \sigma(y)$. Thus, c is a sigma 3-coloring of $G \odot P_n$ using the colors 1, 2, and 3. Therefore, $\nu(G \odot C_n) = 3$. ■

As an immediate consequence of the preceding theorems, we state the following corollary for the corona of a path and a cycle.

Corollary 14. Let P_m be a path and C_n be a cycle of orders $m \geq 4$ and $n \geq 3$, respectively. Then, $\nu(P_m \odot C_n) = \nu(C_n)$.

It is noteworthy that the total color sums of minimal and maximal sigma colorings need not always be different. For example, every sigma 2-coloring of P_2 using the colors 1 and 2 assigns different colors to its two vertices. Hence, its total color sum is always 3, so the minimal and maximal total color sums are equal. Other graphs satisfying this condition are C_3 and the complete bipartite graph $K_{n,n}$. Thus, the condition that the total color sums of a minimal and a maximal sigma coloring are different is not automatically satisfied by every graph.

The next result considers graphs for which such minimal and maximal sigma colorings exist with unequal total color sums and shows that, under this condition, the sigma value is preserved when the graph is used in the corona with a cycle.

Theorem 15. Let $n \geq 4$, and let C_n be a cycle of order n . For $k, r \geq 2$, let G be a nontrivial simple connected graph such that $\sigma(G) = k$ and $\nu(G) = r$. Suppose there exist a minimal sigma k -coloring c' and a maximal sigma k -coloring c'' of G , using the same prescribed set of colors from $\{1, 2, \dots, r\}$, such that $S(c') \neq S(c'')$. Then, $\nu(C_n \odot G) = \nu(G) = r$.

Proof: Let G be a nontrivial simple connected graph such that $\sigma(G) = k$ and $\nu(G) = r$. By definition of sigma value, there exists a sigma coloring of G using k colors from the set $\{1, 2, \dots, r\}$ such that one of the vertices in G is colored r . By assumption, there exist a minimal sigma k -coloring c' and a maximal sigma k -coloring c'' of G , using the same prescribed set of colors, such that $S(c') \neq S(c'')$. Denote these colorings by c' and c'' , respectively. Note that the colors 1 and 2 need not be used by c' and c'' . So, assume that the two smallest colors used in these colorings are a and b , respectively, with $a < b$. By Remark 7, $\sigma(C_n \odot G) = k$ and by Theorem 8, $\nu(C_n \odot G) \leq r$. Now, suppose $\nu(C_n \odot G) < r$. Then, there exists a sigma k -coloring of $C_n \odot G$ with maximum color less than r . Hence, the restricted coloring $c|_G$ is a sigma k -coloring such that the maximum color used is less than r and so, $\nu(G) < r$, a contradiction. We now construct a sigma k -coloring of $C_n \odot G$ using k colors from the set $\{1, 2, \dots, r\}$ as follows:

- (i) If $n \not\equiv 2 \pmod{4}$, then color the vertices v_i in C_n with a when $i \equiv 2 \pmod{4}$, or b , otherwise.
- (ii) If $n \equiv 2 \pmod{4}$, then color the vertices v_i in C_n with a if $i \equiv 3 \pmod{4}$, or b , otherwise.
- (iii) For each of the $n - 1$ copies of G (which are joined to the first $n - 1$ vertices of C_n), color their vertices using the coloring c' , and then color the remaining copy of G (which is joined to the last vertex v_n of C_n) using the coloring c'' .

Note that $\sigma|_{C_n}(u) \in \{a + b, 2b\}$ for any $u \in V(C_n)$. Let $S'_G = \sum_{u \in V(G)} c'(u)$ and $S''_G = \sum_{u \in V(G)} c''(u)$. By definition, for any $x \in V(G)$ and $y \in V(C_n)$, where $xy \in E(C_n \odot G)$, we have $\sigma(x) = \sigma_G(x) + c(y)$ and $\sigma(y) = \sigma_{C_n}(y) + S_G$, where $S_G = S'_G$ or $S_G = S''_G$. We now consider the following cases:

Case 1: Suppose n is even. If $n \equiv 0 \pmod{4}$, then both the restricted colorings $c|_G$ and $c|_{C_n}$ are sigma colorings of G and C_n , respectively. Hence, we are left to show that $\sigma(x) \neq \sigma(y)$, where $x \in V(G)$ and $y \in V(C_n)$ and $xy \in E(C_n \odot G)$. Since $\sigma_G(x) < S_G$, $c(y) \leq b$, and $\sigma_{C_n}(y) \geq a + b$, it follows that $\sigma(x) \leq b + \sigma_G(x) < a + b + S_G \leq \sigma(y)$. Thus, c is a sigma coloring of $\nu(C_n \odot G)$ using k colors from the set $\{1, 2, \dots, r\}$. As a result, $\nu(C_n \odot G) = r = \nu(G)$. On the other hand, suppose $n \equiv 2 \pmod{4}$. Then, in this case, $c|_{C_n}$ is not a sigma coloring of C_n because we will have three consecutive vertices, namely v_1, v_{n-1} , and v_n , with the same color sum $2b$. However, in $C_n \odot G$, we have $\sigma(v_1) = \sigma(v_{n-1}) = 2b + S'_G \neq 2b + S''_G = \sigma(v_n)$. In addition, $\sigma(v_i) = a + b + S'_G$ if i is even and $i \neq n$, and $\sigma(v_i) = 2b + S'_G$ if i is odd. Thus, no two adjacent vertices in C_n have equal color sums in $C_n \odot G$. We also show that $\sigma(x) \neq \sigma(y)$, where $x \in V(G)$ and $y \in V(C_n)$ and $xy \in E(C_n \odot G)$. Since $\sigma_G(x) < S_G$, $c(y) \leq b$ and $\sigma_{C_n}(y) \geq a + b$, it follows that $\sigma(x) \leq b + \sigma_G(x) < a + b + S_G \leq \sigma(y)$. Thus, the coloring described above is also a sigma coloring and that $\nu(C_n \odot G) = \nu(G)$.

Case 2: Suppose n is odd. Analogous to Case 1, $c|_{C_n}$ is not a sigma coloring of C_n . If $n \equiv 1 \pmod{4}$, we have $\sigma|_{C_n}(v_{n-1}) = \sigma|_{C_n}(v_n) = 2b$. Similarly, if $n \equiv 3 \pmod{4}$, then $\sigma|_{C_n}(v_1) = \sigma|_{C_n}(v_n) = a + b$. However, we can use the same argument as in Case 1 to show that $\sigma(v_{n-1}) \neq \sigma(v_n)$ if $n \equiv 1 \pmod{4}$ and $\sigma(v_1) \neq \sigma(v_n)$ if $n \equiv 3 \pmod{4}$ in $C_n \odot G$. ■

Figure 8 shows a sigma 2-coloring of the graph $C_5 \odot P_4$ using the colors 1 and 2. In this coloring, the color sums of the vertices of the first 4 copies of P_n are 1, 2, 3, and 1, respectively, and the color sums of the remaining copy of P_n are 2, 4, 3, and 2, respectively. For each $v_i \in V(C_n)$, $\sigma(v_i) = 8$ if $i \in \{1, 3\}$, $\sigma(v_i) = 9$ if $i \in \{2, 4\}$, and $\sigma(v_5) = 11$.

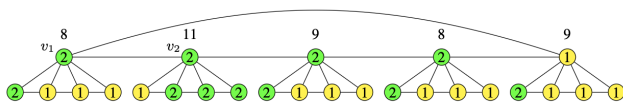


Figure 8: Sigma 2-coloring of $C_5 \odot P_4$

Corollary 16. Let $n \geq 3$ and $m \geq 4$. Then, $\nu(C_n \odot P_m) = 2$ and $\nu(C_n \odot C_m) = \nu(C_m)$.

Corollary 17. Let $n, m \geq 3$. Then, $\nu(C_n \odot K_m) = m$.

In the case of $P_n \odot G$, where $\nu(G) = r \geq 2$ and $G \neq K_n$, we can use an argument similar to that in the proof of Theorem 15 to prove that $\nu(P_n \odot G) = \nu(G)$. In this case, we will color the vertices in P_n using a coloring similar to c_3 if $n \not\equiv 1 \pmod{4}$, or using a coloring c^* if $n \equiv 1 \pmod{4}$. The coloring c^* is a maximal sigma coloring of P_n defined as follows: for all $1 \leq i \leq n$, $c^*(v_i) = a$ if $i \equiv 3 \pmod{4}$ and $c^* = b$, otherwise. Note that $c|_G$ and $c|_{P_n}$ are sigma colorings of G and P_n , respectively. If $x \in V(G)$ and $y \in V(P_n)$, then $\sigma(x) < S_G + c(y) \leq S_G + b \leq S_G + \sigma_{P_n}(y) = \sigma(y)$. Thus, we have the following result.

Theorem 18. Let P_n be a path of order $n \geq 4$, and for $k, r \geq 2$, let G be a nontrivial simple connected graph such that $\sigma(G) = k$ and $\nu(G) = r$. Then, $\nu(P_n \odot G) = \nu(G)$.

Corollary 19. Let $n \geq 2$ and $m \geq 3$. Then, $\nu(P_n \odot K_m) = m$.

Our next two results consider the corona of a cycle C_n with a particular graph whose sigma value is 1.

Theorem 20. Let C_n be a cycle and S_m a star of orders n and m , respectively, where $m, n \geq 3$. Then, $\nu(C_n \odot S_m) = 2$.

Proof: By Remark 1, $\nu(C_n \odot S_m) \neq 1$. To show that $\nu(C_n \odot S_m) = 2$, we construct a sigma coloring of $C_n \odot S_m$ using the colors 1 and 2 as follows:

- (i) Color the vertices of C_n with 1 when the vertices are in even positions and 2 when the vertices are in odd positions.
- (ii) Color the vertices of the first $n - 1$ copies of S_m using only the color 1.
- (iii) If $m > 3$, color the vertices of the last copy of S_m using the colors 1 and 2, where only the internal vertex (the one with degree $m - 1$) is colored 2 and the rest of the vertices are colored 1.
- (iv) If $m = 3$, color all three vertices of the last copy of S_m using only the color 2.

It is straightforward to verify that when n is even, the coloring described above is a sigma coloring of $C_n \odot S_m$. Thus, we will only prove the case when n is odd. Now, note that $c|_{C_n}$ is not a sigma coloring of C_n because $\sigma|_{C_n}(v_1) = \sigma|_{C_n}(v_n) = 3$. However, in $C_n \odot S_m$, we have $\sigma(v_1) = \sigma_{C_n}(v_1) + m \neq \sigma_{C_n}(v_n) + m + 1 = \sigma(v_n)$. As for the other vertices in C_n , we have $\sigma_c(v_i) = 4 + m$ if i is even and $\sigma_c(v_i) = 2 + m$ if i is odd. Thus, no two adjacent vertices in C_n have equal color sums in $C_n \odot S_m$. Now, let $x \in V(S_m)$ and $y \in V(C_n)$ such that $xy \in E(C_n \odot S_m)$. By definition, $\sigma(x) = \sigma_{S_m}(x) + c(y)$ and $\sigma(y) = \sigma_{C_n}(y) + S_m$, where $S_m = \sum_{v \in V(S_m)} c|_{S_m}(v)$. Since $\sigma_{S_m}(x) < S_m$, $\sigma_{C_n}(y) \geq 2$, and $c(y) \in \{1, 2\}$, it follows that $\sigma(x) \leq 2 + \sigma_{S_m}(x) < 2 + S_m \leq \sigma(y)$. Thus, c is a sigma coloring of $\nu(C_n \odot S_m)$ using the colors 1 and 2. As a result, $\nu(C_n \odot S_m) = 2$. ■

In the case of $S_m \odot C_n$, color the vertices in S_m using the color 1 and then use a minimal coloring to color the vertices in C_n using 1 and 2 (respectively, 1, 2, and 3) if n is even (respectively, odd). Hence, we have the following result.

Theorem 21. Let C_n be a cycle of order n , where $n \geq 3$, and let S_m be a star of order m , where $m \geq 3$. Then, $\nu(S_m \odot C_n) = \nu(C_n)$.

CONCLUSION

In this paper, we determined the sigma value of the join of paths and cycles, as well as the corona of a cycle with a star graph. We also provided generalizations for the sigma value of the corona of a path or a cycle with another graph whose sigma value is at least 2. These results extend existing knowledge on sigma coloring and suggest directions for further study on other graph classes and operations. Possible directions for future work include

- (i) determining sigma values for other graph families,
- (ii) investigating the sigma value under other graph operations, and
- (iii) characterizing pairs of graphs G and H for which $\nu(G \odot H) = \nu(H)$, by identifying coloring conditions on G and H that guarantee this equality.

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CONFLICT OF INTEREST

The authors declare that there is no conflict of interest.

CONTRIBUTIONS OF INDIVIDUAL AUTHORS

The research concept and overall study structure were jointly developed by the authors, with emphasis on sigma values of graph families under join and corona operations. Mahinay and Lagura-Arreza established the initial results, which were reviewed and verified by Garciano and Marcelo. Lagura-Arreza was responsible for preparing and writing the manuscript, including organizing the discussion and refining the article title to reflect the scope and contributions of the study.

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